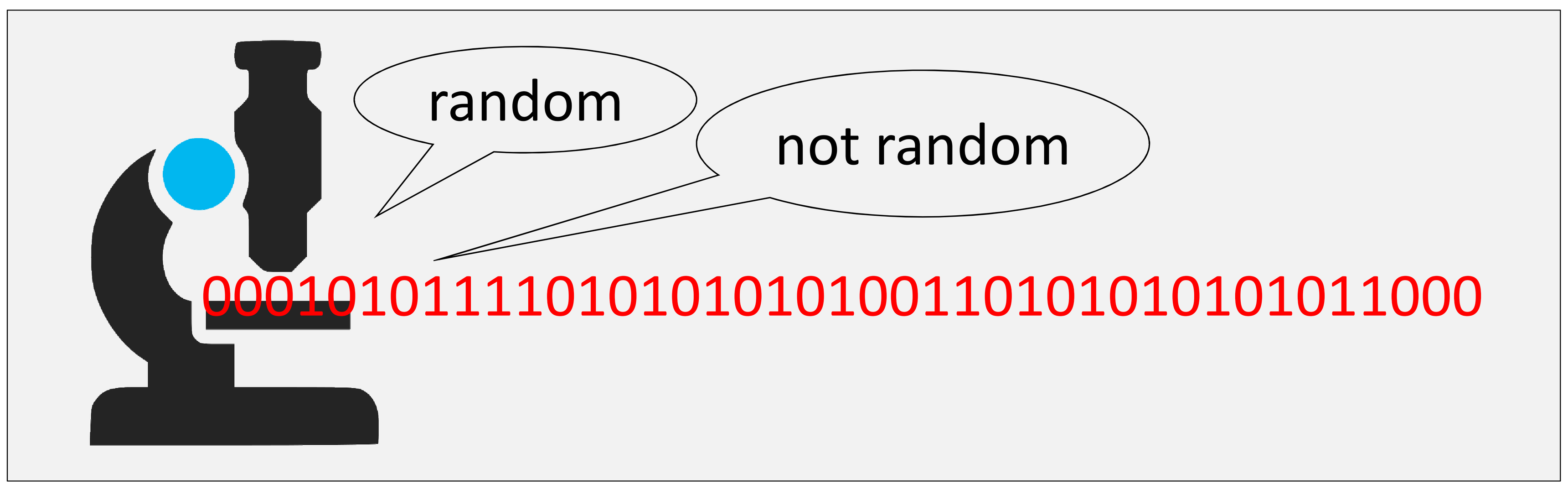
Bounded independence vs. moduli

Ravi Boppana, Johan Håstad, Chin Ho Lee and Emanuele Viola

Pseudorandomness

- Given a string sampled from a distribution D
- Can you test if it comes from D or it is random?



A distribution D **fools** a test T if
$|\Pr[T(D)\ accepts] - \Pr[T(U)\ accepts]| \le 1/3$,
where U is the uniform distribution.

What are mod m tests?

- Count the number of 1s in the input string
- Check if it is divisible by m

A **mod m test** on n bits accepts if
the number of 1's in the input is divisible by m.

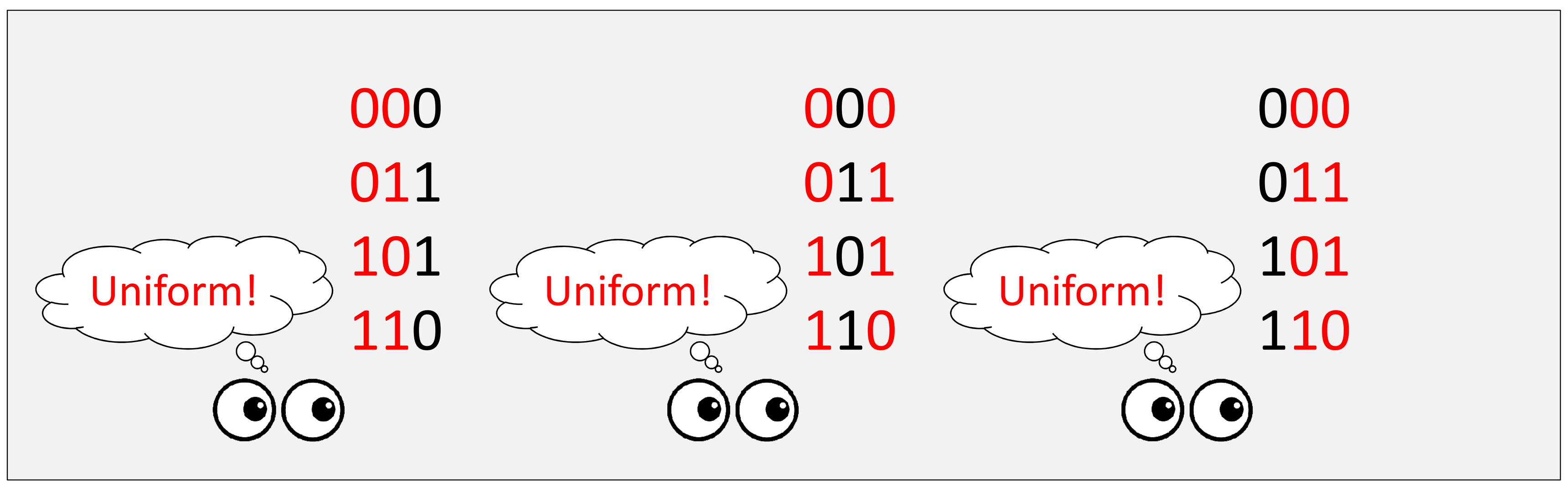
What are k-wise uniform distributions on n bits?

- Look at any of the k bits of the distribution
- These k bits must be uniformly distributed

A distribution D on n bits is **k-wise uniform** if
its marginal distribution on every k bits is uniform.

Example: a 2-wise uniform distribution on 3 bits

Sample a string from $\{000, 011, 101, 110\}$ at random



These strings have the same parity

What can k-wise uniform distributions fool?

- Any test on k bits (by definition)
- Combinatorial rectangles, low-depth circuits, halfspaces, etc.

For what values of k, every k-wise uniform distribution fools mod *m test?*

Fails completely when $m = 2, k = n - 1$

- Look at our example
- All the strings in the distribution are accepted by mod 2 test!

What about m = 3?

- What is the largest k such that there exists a k-wise distribution in which all strings are accepted by mod 3 test?
- Somewhat surprisingly, k can still be $\Omega(n)$!

Our results

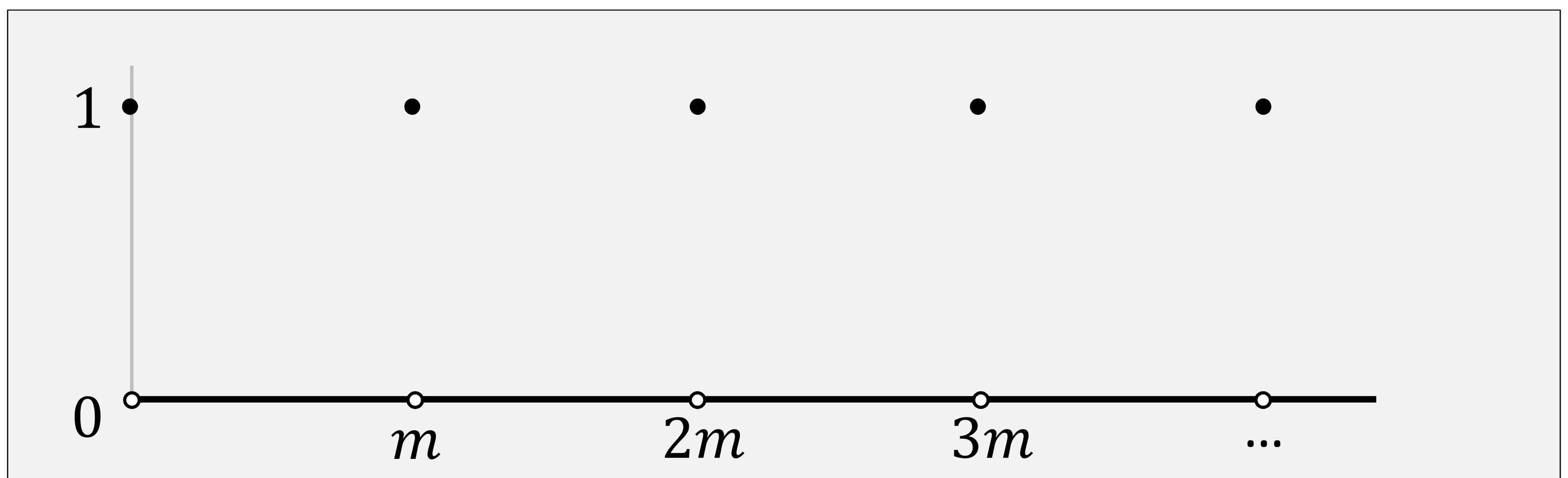
If $k = \Omega(n/m)$ then every k-wise uniform distribution fools mod *m test.*

If $k = O(n/m^2 \log m)$ then some k-wise uniform distribution fails to fool mod *m test.*

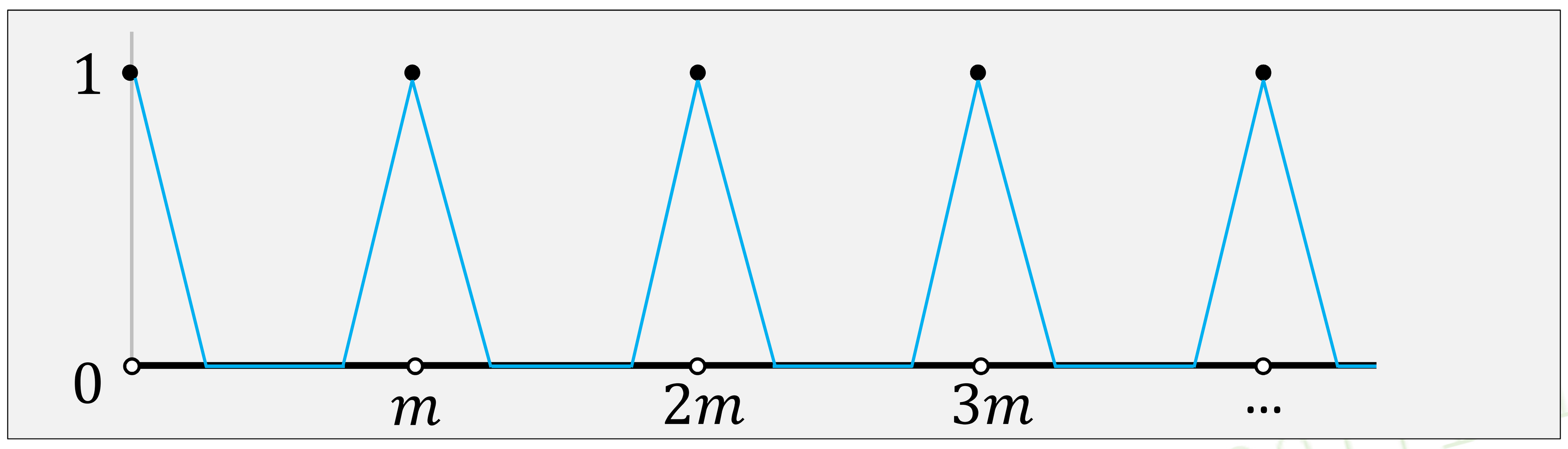
Techniques

Fourier analysis, approximation theory, etc.

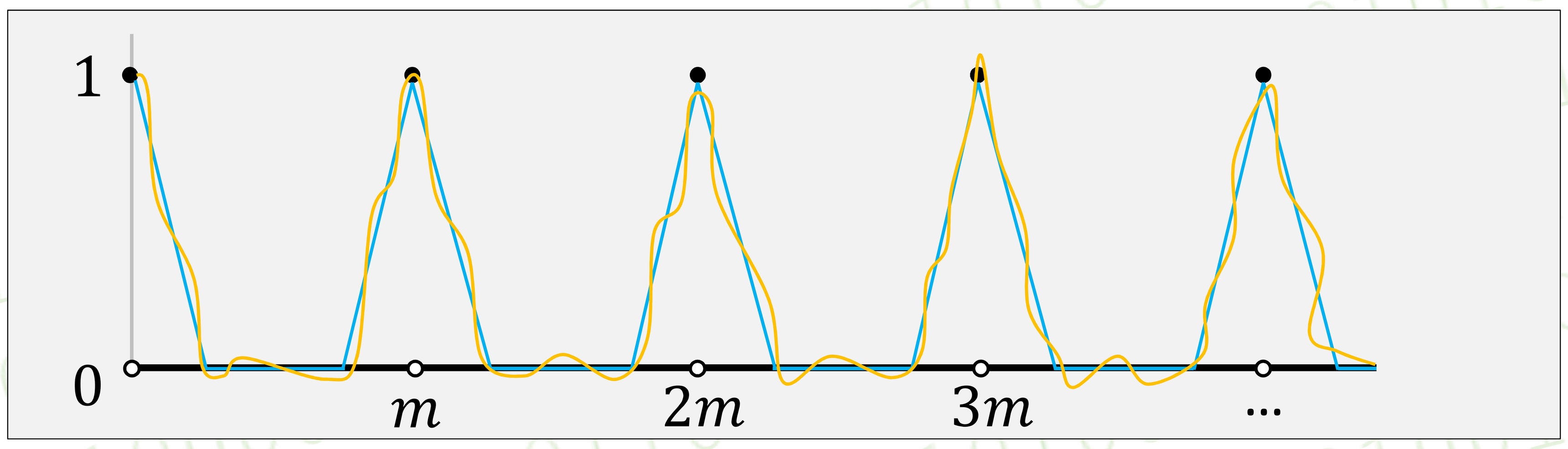
Approximation theory



Symmetrization



Continuous approximation



Low-degree approximation